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GEOMETRY OF HIGHER ORDER DIFFERENTIAL EQUATIONS OF FINITE TYPE ASSOCIATED WITH SYMMETRIC SPACES

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§1. Differential Equations of Finite Type

We will consider Higher order ODE

$$y^{(k)} = F(x, y, y', \ldots, y^{(k-1)}).$$

Or more generally

$$\frac{\partial^k y}{\partial x_{i_1} \cdots \partial x_{i_k}} = F_{i_1 \cdots i_k}(x_1, \ldots, x_n, y, p_1, \ldots, p_n, \ldots, p_{i_1 \cdots i_{k-1}}) \quad (1 \leqq i_1 \leqq \cdots \leqq i_k \leqq n),$$

where $p_{i_1 \cdots i_\ell} = \frac{\partial^\ell y}{\partial x_{i_1} \cdots \partial x_{i_\ell}}$. These equations define a submaifold R in k-jets space J^k.

$$J^k \supset R \to J^{k-1}; \text{Diffeomorphism} \quad (*)$$

We have the Contact system C^k on J^k

$$\begin{cases} \varpi = dy - \sum p_i dx_i, \\ \varpi_i = dp_i - \sum p_{ij} dx_j, \\ \cdots\cdots\cdots\cdots, \\ \varpi_{i_1 \cdots i_{k-1}} = dp_{i_1 \cdots i_{k-1}} - \sum p_{i_1 \cdots i_{k-1} j} dx_j. \end{cases}$$

C^k gives a foliation on R when R is integrable.

Through the diffeomorphism $(*)$, R defines a differential system E on J^{k-1} such that

$$C^{k-1} = E \oplus F, \quad F = \operatorname{Ker}(\pi^{k-1}_{k-2})_*$$

where $\pi^{k-1}_{k-2} : J^{k-1} \to J^{k-2}$ is the bundle projection.

N.Tanaka introduced the notion of pseudo-product manifolds as follows.

Pseudo-Product Manifolds $(R; E, F)$

(1) $E \cap F = 0$, and both E and F are completely integrable.

(2) $D = E \oplus F$ is non-degenerate.

(3) The full derived systems of D coincides with $T(R)$

N.Tanaka, *On affine symmetric spaces and the automorphism groups of product manifolds,* Hokkaido Math.J. **14** (1985), 277-351

§2. Geometry of Linear Differential Systems (Tanaka Theory)

We summarize here the basic notion for differential systems.

For a manifold M of dimension d, a subbundle $D \subset T(M)$ of rank r $(s + r = d)$ is called a differential system of rank r (or codimension s).

$$D = \{ \omega_1 = \cdots = \omega_s = 0 \}.$$

(M, D) is completely integrable
$\Longleftrightarrow D = \{dx_1 = \cdots = dx_s = 0\}$
$\Longleftrightarrow d\omega_i \equiv 0 \pmod{\omega_1, \cdot\cdot, \omega_s}$ $(1 \leqq i \leqq s)$
$\Longleftrightarrow [\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$ where $\mathcal{D} = \Gamma(D)$

For non-completely integrable system, we have
Derived System ∂D: $\partial\mathcal{D} = \mathcal{D} + [\mathcal{D}, \mathcal{D}]$.
Cauchy Characteristic System $Ch(D)$:

$$Ch(D)(x) = \{X \in D(x) \mid X \rfloor d\omega_i \equiv 0 \pmod{\omega_1, \ldots, \omega_s} \quad \text{for } i = 1, \ldots, s \},$$

k-th **Derived System** $\partial^k D$:

$$\partial^k D = \partial(\partial^{k-1} D)$$

k-th **Weak Derived System** $\partial^{(k)} D$:

$$\partial^{(k)}\mathcal{D} = \partial^{(k-1)}\mathcal{D} + [\mathcal{D}, \partial^{(k-1)}\mathcal{D}],$$

Symbol Algebras
(M, D) is called **regular** iff
$(S1)$ $\exists \mu > 0$ such that, for all $k \geqq \mu$,

$$D^{-k} = \cdots = D^{-\mu} \supsetneqq \cdots \supsetneqq D^{-2} \supsetneqq D^{-1} = D,$$

$(S2)$ $[\mathcal{D}^p, \mathcal{D}^q] \subset \mathcal{D}^{p+q}$ for all $p, q < 0$.

From now on, we will consider **regular** diffrential systems (M, D) such that $T(M) = D^{-\mu}$.

Symbol algebra $\mathfrak{m}(x)$ of (M, D) at x is defined as follows;
$\forall x \in M$,

$$\mathfrak{m}(x) = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x).$$

$\mathfrak{g}_{-1}(x) = D^{-1}(x)$, $\mathfrak{g}_p(x) = D^p(x)/D^{p+1}(x)$

$$[X, Y] = \varpi_{p+q}([\tilde{X}, \tilde{Y}]_x),$$

$$\begin{cases} \tilde{X} \in \Gamma(D^p), X = \varpi_p(\tilde{X}_x) \in \mathfrak{g}_p(x), \\ \tilde{Y} \in \Gamma(D^q), Y = \varpi_q(\tilde{Y}_x) \in \mathfrak{g}_q(x). \end{cases}$$

$$\mathfrak{g}_p(x) = [\mathfrak{g}_{p+1}(x), \mathfrak{g}_{-1}(x)] \qquad \text{for } p < -1.$$

Conversely given a

Fundamental Graded Lie Algebra :

$$\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$$

i.e., Nilpotent GLA satisfying the generating condition :

$$\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}] \quad \text{for } p < -1$$

We have the notions of

Standard Differential System $(M(\mathfrak{m}), D_{\mathfrak{m}})$ of type $\mathfrak{m}$

Prolongation $\mathfrak{g}(\mathfrak{m})$ of $\mathfrak{m} = \oplus_{p<0}\mathfrak{g}_p$

N.Tanaka, *On differential systems, graded Lie algebras and pseudo-groups,* J.Math. Kyoto Univ. **10** (1970), 1-82

§3. Symbol Algebra of (J^k, C^k)

$$C^k = \{\varpi = \varpi_i = \cdots = \varpi_{i_1\cdots i_{k-1}} = 0\}$$

$$\mathfrak{C}^k(m,n) = \mathfrak{C}_{-(k+1)} \oplus \mathfrak{C}_{-k} \oplus \cdots \oplus \mathfrak{C}_{-1}$$

where $\mathfrak{C}_{-(k+1)} = W, \mathfrak{C}_p = W \otimes S^{k+p+1}(V^*)$, $\mathfrak{C}_{-1} = V \oplus W \otimes S^k(V^*)$.

Coframe:

$$\{\varpi, \ldots, \varpi_{i_1\cdots i_\ell}, \ldots, dx_i, dp_{i_1\cdots i_k}\},$$

Dual Frame:

$$\{\frac{\partial}{\partial y}, \ldots, \frac{\partial}{\partial p_{i_1\cdots i_\ell}}, \ldots, \frac{d}{dx_i}, \frac{\partial}{\partial p_{i_1\cdots i_k}}\}$$

where

$$\frac{d}{dx_i} = \frac{\partial}{\partial x_i} + \sum p_{ij_1\cdots j_\ell}\frac{\partial}{\partial p_{j_1\cdots j_\ell}}$$

K.Yamaguchi, *Contact Geometry of Higher Order,* Japan. J. Math.**8** (1982), 109-176

Pseudo-projective strucures of order k of bidegree (m,n)

Starting from

$$\mathfrak{C}_{-1} = \mathfrak{e} \oplus \mathfrak{f},$$

where $\mathfrak{e} = V, \mathfrak{f} = W \otimes S^{k-1}(V^*)$.

This splitting represents the pseudo-product structure of k-th order equation R in the symbol level.

Put

$$\check{\mathfrak{g}}_0 = \{X \in \mathrm{Der}_0(\mathfrak{C}^{k-1}(m,n)) : [X, \mathfrak{e}] \subset \mathfrak{e}, [X, \mathfrak{f}] \subset \mathfrak{f}\}$$

Peudo-projective GLA

$$\mathfrak{g}^k(m,n) = \text{Prolongation of } (\mathfrak{C}^{k-1}(m.n), \check{\mathfrak{g}}_0)$$

Cartan Connections

T.Morimoto, *Geometric structures on filtered manifolds,* Hokkaido Math. J. **22**(1993), 263-347

§4. Pseudo-product GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$

Now we will give the notion of the pseudo-product GLA of type $(\mathfrak{l}, S)$.

$\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$: **reductive GLA**

(1) $\hat{\mathfrak{l}} = \mathfrak{l}_{-1} \oplus [\mathfrak{l}_{-1}, \mathfrak{l}_1] \oplus \mathfrak{l}_1$ is simple.

(2) $\mathfrak{z}(\mathfrak{l}) \subset \mathfrak{l}_0$.

S: **faithful irreducible $\mathfrak{l}$ -module.**

$$S_{-1} = \{s \in S : \mathfrak{l}_1 \cdot s = 0\}$$

$$S_p = \mathrm{ad}(\mathfrak{l}_{-1})^{-p-1} S_{-1} \quad \text{for } p < 0$$

Form the semi-direct product

$$\mathfrak{g} = S \oplus \mathfrak{l}, \qquad [S, S] = 0$$
$$\mathfrak{g}_p = \mathfrak{l}_p \ (p \geqq 0), \quad \mathfrak{g}_{-1} = \mathfrak{l}_{-1} \oplus S_{-1},$$
$$\mathfrak{g}_q = S_q \ (q \leqq -2).$$

Then the following hold:

(1) $S = \bigoplus_{p=-1}^{-\mu} S_p$;
(2) $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is generated by $\mathfrak{g}_{-1}$
(3) $S_{-\mu} = \{s \in S : [\mathfrak{l}_{-1}, s] = 0\}$
(4) $S_p \hookrightarrow W \otimes S^{\mu+p}(\mathfrak{l}_{-1}{}^*), \quad W = S_{-\mu}$
(5) S_{-1}, $S_{-\mu}$: irreducible $\mathfrak{l}_0$ -modules

Thus, $\mathfrak{m}$ is a symbol algebra of μ-th order differential equations of finite type.

We will ask the following questions.

Our Problem

(1) When is $\mathfrak{g}$ the prolongation of $\mathfrak{m}$ or $(\mathfrak{m}, \mathfrak{g}_0)$?

(2) Find the fundamental invariants for equations of type $\mathfrak{m}$.

§5. Y. Se-ashi's Theory for Linear Differential Equations of Finite Type

For the linear differential equations of finite type, we have the following theory due to Y. Se-ashi.

Y.Se-ashi, *On differential invariants of integrable finite type linear differential equations,* Hokkaido Math.J., **17** (1988), 151-195

In particular, he established the **Rigidity Theorem** of Equations of type $(\mathfrak{l}, S)$ for $M = L/L'$ other than projective spaces and quadrics

Utilizing this Rigidity Theorem, we have an **Application to Hypergeometric Equations**

T. Sasaki, K. Yamaguchi and M. Yoshida, *On the Rigidity of Differential Systems modelled on Hermitian Symmetric Spaces and Disproofs of a Conjecture concerning Modular Interpretations of Configuration Spaces,* Advanced Studies in Pure Math. 25 (1997), 318-354

§6. Generalized Spencer cohomology

$\mathfrak{a} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{a}_p$: graded Lie algebra
$V = \bigoplus_{p\in\mathbb{Z}} V_p$: graded $\mathfrak{a}$-module

Cohomology space $H^q(\mathfrak{a}, V)$ associated with $(C^q(\mathfrak{a}, V), \partial) = (\mathrm{Hom}(\bigwedge^q \mathfrak{a}, V), \partial)$ here $\partial^q : C^q(\mathfrak{a}, V) \to C^{q+1}(\mathfrak{a}, V)$ is given by

$$\partial^q\omega(x_1, \ldots, x_{q+1}) = \textstyle\sum_{i=1}^{q+1}(-1)^{i+1}x_i \cdot \omega(x_1, \ldots, \hat{x}_i, \ldots, x_{q+1}) \\ + \displaystyle\sum_{i<j}(-1)^{i+j}\omega([x_i, x_j], x_1, \cdot\cdot, \hat{x}_i, \cdot\cdot, \hat{x}_j, \cdot\cdot, x_{q+1}),$$

where $x_i \in \mathfrak{a}$ and $\omega \in C^q(\mathfrak{a}, V)$.
Moreover

$$C^q(\mathfrak{a}, V) = \bigoplus_r C^q(\mathfrak{a}, V)_r,$$

$$C^q(\mathfrak{a}, V)_r = \{\, \omega \in C^q(\mathfrak{a}, V) : \omega(\mathfrak{a}_{i_1} \wedge \cdots \wedge \mathfrak{a}_{i_q}) \subset V_{i_1+\cdots+i_q+r} \,\}.$$

Cohomology group $H^*(\mathfrak{m}, \mathfrak{g})$ associated with the adjoint representation of $\mathfrak{m}$ on $\mathfrak{g}$

Put $\mathfrak{b}_{-1} = S$, $\mathfrak{b}_0 = \mathfrak{l}$, $\mathfrak{b}_p = 0$ $(p \neq -1, 0)$

$$\mathfrak{g} = \bigoplus_p \mathfrak{b}_p = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0, \quad \mathfrak{b}_- = \mathfrak{b}_{-1}$$

Cohomology group $H^*(\mathfrak{b}_-, \mathfrak{g})$ associated with the adjoint rep. of $\mathfrak{b}_-$ on $\mathfrak{g}$

$H^i(\mathfrak{b}_-, \mathfrak{g})$ is a $\mathfrak{l}$-module

Theorem 6.1.
$H^q(\mathfrak{m},\mathfrak{g}) \cong \bigoplus_{i=0}^q H^{q-i}(\mathfrak{l}_-, H^i(\mathfrak{b}_-,\mathfrak{g}))$ *as a* $\mathfrak{l}_0$*-module.*

Explicitly for $q=1$

$$H^1(\mathfrak{m},\mathfrak{g}) \cong H^1(\mathfrak{l}_-, S) \oplus H^0(\mathfrak{l}_-, S\otimes S^*/\mathfrak{l}) \oplus H^0(\mathfrak{l}_-, \check{\mathfrak{b}}_1)$$

where $\check{\mathfrak{b}} = \bigoplus_{p\in\mathbb{Z}} \check{\mathfrak{b}}_p$ *is the prolongation of* $\mathfrak{g} = \mathfrak{b}_{-1}\oplus\mathfrak{b}_0 = S\oplus\mathfrak{l}$

Gradations of cohomology groups

Put $\mathfrak{g}_{p,q} = \mathfrak{g}_p \cap \mathfrak{b}_q$

$$C^q(\mathfrak{m},\mathfrak{g})_{r,s} = \{\omega \in \mathrm{Hom}(\textstyle\bigwedge^q \mathfrak{m},\mathfrak{g}) : \omega(\mathfrak{g}_{i_1,j_1}\wedge\cdots\wedge\mathfrak{g}_{i_q,j_q}) \subset \mathfrak{g}_{i_1+\cdots+i_q+r, j_1+\cdots+j_q+s} \text{for all } i_1,\ldots,i_q,j_1,\ldots,j_q\}$$

$$H^*(\mathfrak{m},\mathfrak{g}) = \bigoplus_{q,r,s} H^q(\mathfrak{m},\mathfrak{g})_{r,s}$$

$$C^q(\mathfrak{b}_-,\mathfrak{g})_s = \{\omega \in \mathrm{Hom}(\textstyle\bigwedge^q \mathfrak{b}_-,\mathfrak{g}) : \omega(\mathfrak{b}_{j_1}\wedge\cdots\wedge\mathfrak{b}_{j_q}) \subset \mathfrak{b}_{j_1+\cdots+j_q+s} \text{for all } j_1,\ldots,j_q<0\}$$

$$H^*(\mathfrak{b}_-,\mathfrak{g}) = \bigoplus_{q,s} H^q(\mathfrak{b}_-,\mathfrak{g})_s$$

Thus

$$H^q(\mathfrak{m},\mathfrak{g})_{r,s} \cong \bigoplus_{i=0}^{q} H^{q-i}(\mathfrak{l}_-, H^i(\mathfrak{b}_-,\mathfrak{g})_s)_r$$

Utilizing the following theorems

Theorem A (Kostant)*Let* $\mathfrak{s} = \bigoplus_{p\in\mathbb{Z}}\mathfrak{s}_p$ *be a simple GLA of type* (X_l,Δ_1) *and* $M(\omega)$ *be an irreducible* $\mathfrak{s}$*-module with lowest weight* ω*. Then*

$$\mathrm{ch}_{\mathfrak{s}_0}(H^j(\mathfrak{s}_-, M(\omega))) = \sum_{w\in W_1^j} \mathrm{ch}_{\mathfrak{s}_0}(m(w(\omega-\rho)+\rho)),$$

where ρ *is the half sum of positive roots.*

Theorem B (Kobayashi-Nagano)
S*: faithful irreducible* $\mathfrak{l}$*-module.*

If $\check{\mathfrak{b}}_1 \neq \{0\}$,

(1) $\dim\check{\mathfrak{b}} < \infty$
$\check{\mathfrak{b}} = \mathfrak{b}_{-1}\oplus\mathfrak{b}_0\oplus\check{\mathfrak{b}}_1$*: simple*

$$\mathfrak{b}_{-1} = S, \quad \mathfrak{b}_0 = \mathfrak{l}, \quad \check{\mathfrak{b}}_1 = S^*$$

(2) $\dim\check{\mathfrak{b}} = \infty$
$\mathfrak{l} = \mathfrak{gl}(S)$ or $\mathfrak{csp}\ (S)$

We have the following answer for our problem (1) cited in §4.

Theorem 6.2. *Let* $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *be a pseudo-product GLA of type* $(\mathfrak{l}, S)$. *Except for* (1), (2), (3),

$$\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p \cong \mathfrak{g}(\mathfrak{m}),$$

where $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$

(1) $0 < \dim \check{\mathfrak{b}} < \infty$ ($\check{\mathfrak{b}}$: *simple*)

$\mathcal{D}(\mathfrak{l})$	λ	$\check{\mathfrak{b}} = (Y_{\ell+1}, \Sigma_1)$
$(A_i \times A_{\ell-i}, \{\alpha_j\})$ $(j \leqq i)$	$\varpi_1 + \varpi_i$	$(A_{\ell+1}, \{\alpha_j, \alpha_{i+1}\})$
$(B_\ell, \{\alpha_1\})$ $(\ell \geqq 3)$	ϖ_1	$(B_{\ell+1}, \{\alpha_1, \alpha_2\})$
$(A_\ell, \{\alpha_i\})$ $(\ell \geqq 2)$	$2\varpi_\ell$	$(C_{\ell+1}, \{\alpha_i, \alpha_{\ell+1}\})$
$(D_\ell, \{\alpha_\ell\})$ $(\ell \geqq 4)$	ϖ_1	$(D_{\ell+1}, \{\alpha_1, \alpha_{\ell+1}\})$
$(D_\ell, \{\alpha_1\})$ $(\ell \geqq 4)$	ϖ_1	$(D_{\ell+1}, \{\alpha_1, \alpha_2\})$
$(A_\ell, \{\alpha_i\})$ $(\ell \geqq 4)$ $(1 < i < \ell)$	$\varpi_{\ell-1}$	$(D_{\ell+1}, \{\alpha_i, \alpha_{\ell+1}\})$
$(D_5, \{\alpha_5\})$	ϖ_5	$(E_6, \{\alpha_1, \alpha_3\})$
$(D_5, \{\alpha_4\})$	ϖ_5	$(E_6, \{\alpha_1, \alpha_2\})$
$(D_5, \{\alpha_1\})$	ϖ_5	$(E_6, \{\alpha_1, \alpha_6\})$
$(E_6, \{\alpha_1\})$	ϖ_6	$(E_7, \{\alpha_1, \alpha_7\})$
$(E_6, \{\alpha_6\})$	ϖ_6	$(E_7, \{\alpha_6, \alpha_7\})$

(2) $\dim \check{\mathfrak{b}} = \infty$

$\mathcal{D}(\mathfrak{l})$	λ	$\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$
$(A_\ell, \{\alpha_i\})$	ϖ_1	$(A_{\ell+1}, \{\alpha_i, \alpha_{\ell+1}\})$
$(C_\ell, \{\alpha_\ell\})$	ϖ_1	$\mathfrak{g}$

In $(C_\ell, \{\alpha_\ell\})$*-case,* $\mu = 2$

$$S_{-2} = V^*, \quad S_{-1} = V, \quad \mathfrak{l}_{-1} = S^2(V^*),$$
$$\mathfrak{l}_0 = V \otimes V^* \oplus \mathbb{C}, \qquad \mathfrak{l}_1 = S^2(V)$$

(3) $\mathfrak{g}$ *is a pseudo-projective GLA, i.e.,* $\mathcal{D}(\mathfrak{l}) = (A_\ell \times A_n, \{\alpha_1\})$ *and* $\lambda = k\varpi_1 + \pi_1$ $\mu = k + 1$ *and* $\dim W = n + 1$

$$S_{-\mu} = W, \quad S_p = W \otimes S^{\mu+p}(V^*) \ (-\mu < p < 0),$$
$$\mathfrak{l}_{-1} = V, \quad \mathfrak{l}_0 = \mathfrak{gl}(V) \oplus \mathfrak{sl}(W), \quad \mathfrak{l}_1 = V^*$$

§7. Gradation of Simple Lie Algebras.

We summarize here the relavent terminology to express the contents of Theorem 6.2.

$\mathfrak{s}$: Simple Lie Algebra over $\mathbb{C}$
$\mathfrak{h}$: Cartan Subalgebra ; $\Phi \subset \mathfrak{h}^*$: Root System
$\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$: Simple Root System

$$\mathfrak{s} = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_{-\alpha},$$

$\Delta_1 \subset \Delta$: Fix, $\quad \Phi^+ = \cup_{p\geqq 0}\Phi_p^+,$

$$\Phi_p^+ = \{\alpha = \sum_{i=1}^{\ell} n_i\alpha_i \mid \sum_{\alpha_i\in\Delta_1} n_i = p\},$$

$$\begin{cases} \mathfrak{s}_p = \bigoplus_{\alpha\in\Phi_p^+} \mathfrak{g}_\alpha, & (p > 0) \\ \mathfrak{s}_0 = \bigoplus_{\alpha\in\Phi_0^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi_0^+} \mathfrak{g}_{-\alpha}, \\ \mathfrak{s}_{-p} = \bigoplus_{\alpha\in\Phi_p^+} \mathfrak{g}_{-\alpha}, \end{cases}$$

Then

$$[\mathfrak{s}_p, \mathfrak{s}_q] \subset \mathfrak{s}_{p+q} \qquad \text{for} \quad p, q \in \mathbb{Z}.$$

Generating Condition: $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{s}_p$

$$(\star) \quad \mathfrak{s}_p = [\mathfrak{s}_{p+1}, \mathfrak{s}_{-1}] \quad \text{for} \quad p < -1$$

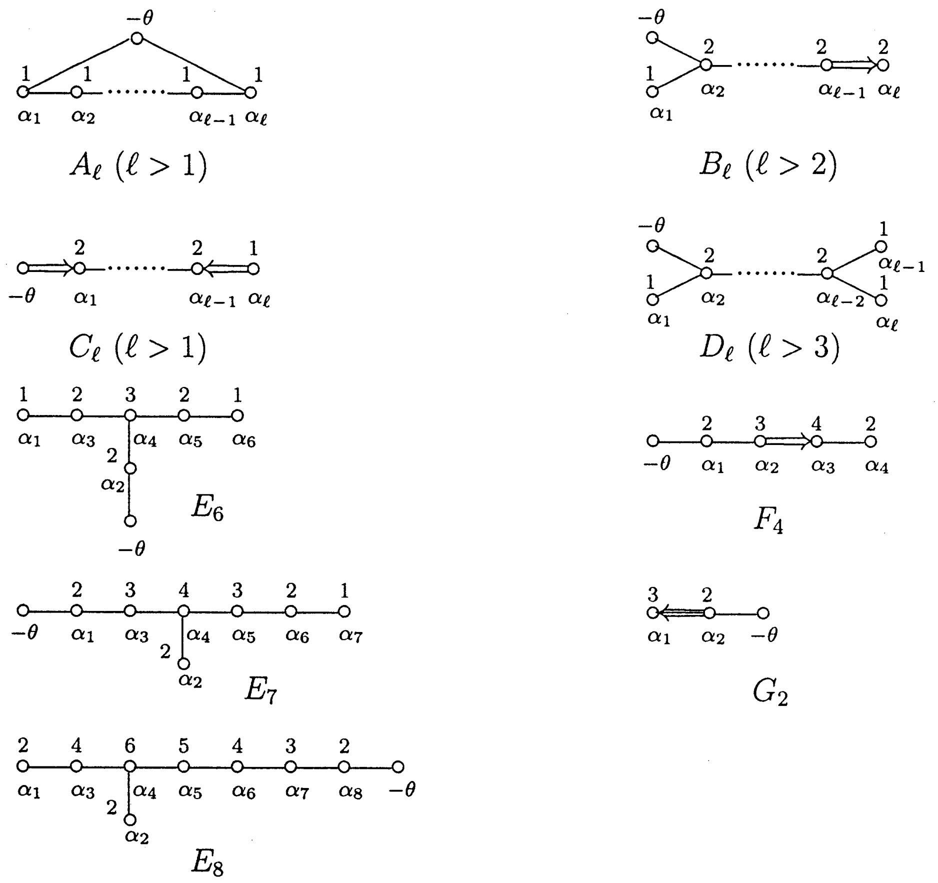
$$\Delta_1 \subset \Delta \quad \Longrightarrow \quad (X_\ell, \Delta_1): \quad \mathfrak{s} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{s}_p$$

where $\mu = \sum_{\alpha_i\in\Delta_1} n_i(\theta)$, $\theta = \sum_{i=1}^{\ell} n_i(\theta)\,\alpha_i$

Theorem 7.1. $\mathfrak{s} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{s}_p$:*Simple Graded Lie Algebra over* $\mathbb{C}$ *satisfying* $(\star)$.
X_ℓ: *Dynkin Diagram of* $\mathfrak{s}$.
$\Longrightarrow$
$\exists_1\ \Delta_1 \subset \Delta$ *s.t.* $\mathfrak{s} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{s}_p \cong (X_\ell, \Delta_1)$

Classification of $\mathfrak{s} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{s}_p$ **with** $(\star)$ is equivalent to

Classification of Parabolic subalgebras $\mathfrak{s}' = \bigoplus_{p\geqq 0} \mathfrak{s}_p$



Extended Dynkin Diagrams
with the coefficient of the highest root

§8. Second Cohomology

We summarize here the results on the second cohomology.

$$H^2(\mathfrak{m}, \mathfrak{g})_{r,-1} \cong H^2(\mathfrak{l}_-, \mathfrak{b}_{-1})_r,$$

Proposition 8.1.

(1) $H^2(\mathfrak{m}, \mathfrak{g})_{r,-1} = 0$ *for all* $r \geq 2$.
(2) $H^2(\mathfrak{m}, \mathfrak{g})_{1,-1} \neq 0$ *iff the sequence* $(X_\ell, \Delta_1, \lambda)$ *is one of the following* $(A_\ell, \{\alpha_1\}, j\varpi_{\ell-1} + k\varpi_\ell)(\ell \geqq 2, j, k \geqq 0, j+k \geqq 1)$, $(A_\ell, \{\alpha_2\}, k\varpi_\ell)$ $(\ell \geqq 3, k \geqq 1)$, $(C_2, \{\alpha_2\}, k\varpi_1)$ $(k \geqq 1)$

$$H^2(\mathfrak{m},\mathfrak{g})_{r,0} \cong H^1(\mathfrak{l}_-, H^1(\mathfrak{b},\mathfrak{g})_0)_r$$

Proposition 8.2.

(1) $H^2(\mathfrak{m},\mathfrak{g})_{r,0} = 0$ *for* $r \geqq 2$ *except for the case when* $(X_\ell, \Delta_1) = (A_\ell, \{\alpha_1\})$ *or* $(A_\ell, \{\alpha_\ell\})$.

(2) $H^2(\mathfrak{m},\mathfrak{g})_{1,0} = 0$ *if* (X_ℓ, Δ_1) *is one of* $(A_\ell, \{\alpha_i\})$ $(\ell \geqq 4,\ 1 < i \leqq [\frac{l+1}{2}])$, $(C_\ell, \{\alpha_\ell\})$ $(\ell \geqq 3)$, $(D_\ell, \{\alpha_{\ell-1}\})$ $(\ell \geqq 5)$, $(E_6, \{\alpha_1\})$, $(E_7, \{\alpha_7\})$.

(3) *If* $(X_\ell, \Delta_1) = (A_\ell, \{\alpha_1\})$, *then*
$H^2(\mathfrak{m},\mathfrak{g})_{r,0} = 0$ *for* $r \geqq \min\{m_1, m_l\} + 2$.

$$H^2(\mathfrak{m},\mathfrak{g})_{r,1} \cong H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-,\mathfrak{g})_1)_r \oplus H^1(\mathfrak{l}_-, \check{\mathfrak{b}}_1)_r,$$

$$H^2(\mathfrak{m},\mathfrak{g})_{r,2} \cong H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-,\mathfrak{g})_2)_r$$

Proposition 8.3.

(1) *For* $s = 1, 2$,
$$H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-,\mathfrak{g})_s)_r = 0 \qquad \text{for } r \geqq s(\mu - 1) + 1,$$

(2) *If* $X_l = B_l$, C_l *or* E_7, *then*
$$H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-,\mathfrak{g})_s)_r = 0 \qquad \text{for } r \geqq [s(\mu + 1)/2] + 1.$$

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